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A note on the free convection in a wall plume: horizontal wall effects

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1. INTRODUCTION

A PLUME arising from a heated element is a problem of considerable interest in several engineering applications, e.g. hot-wire anemometry, flows that arise in fire studies, cooling of electronic circuitry, meteorology, discharge of technological waste, etc. There has been a substantial amount of work performed on free plumes from a line or a point thermal source over the last few years. A comprehensive review of this work has been given recently by Gebhart *et al.* [1].

The flow resulting from a horizontal line source of heat embedded at the base of a vertical adiabatic surface is conventionally referred to as the plane wall plume. Problems of this type have been investigated by Zimin and Lyakhov [2], Liburdy and Faeth [3], Jaluria and Gebhart [4], Afzal [5], Rao *et al.* [6], Krishnamurthy and Gebhart [7], Mörwald *et al.* [8], Lin and Chen [9] and Joshi [10].

The purpose of the present Note is to re-examine the development of a laminar free convection plume rising from a steady line thermal source of heat embedded at the leading edge of an adiabatic vertical surface bounded by an insulated horizontal wall, which is placed at the level of the heat source, for moderately large values of the Grashof number. The flow field is divided into three distinct regions: the bulk flow, in which up to fourth-order correction the flow is potential, and two inner boundary layer regions. The method of matched asymptotic expansions is used to obtain a consistent solution by *simultaneously* including the effects of both inner boundary layers and of higher-order boundary layer corrections. The need for such a treatment is demonstrated by the recent work of Mörwald *et al.* [8], Thomas and Takhar [11] and Ingham and Pop [12]. It is shown that the presence of the horizontal surface exerts quite a substantial influence on the higher-order approximations in the plume boundary layer and on the outer flow region.

To the authors' knowledge, no experimental investigation of the problem considered here has been reported in the literature.

2. ANALYSIS

The geometry considered in this study is equivalent to a line thermal source of heat embedded at the leading edge of an adiabatic vertical plane surface bounded by a horizontal wall, which is placed at the level of the heat source and is maintained at the temperature T_∞ of the ambient fluid as shown in Fig. 1. The Cartesian coordinate system $(\hat{x}, \hat{y})$ with the origin at the leading edge of the vertical plate is oriented so that the positive $\hat{x}$-axis is along the vertical plate in the upward direction and the $\hat{y}$-axis is perpendicular to it. For this two-dimensional flow geometry the governing equations, using the Boussinesq approximation, were written in convenient non-dimensional form by Riley [13] as

$$\frac{\partial(\psi, \nabla^2\psi)}{\partial(x,y)} + \frac{\partial\Phi}{\partial y} + Gr^{-2/5}\nabla^4\psi = 0 \qquad (1)$$

$$\frac{\partial(\psi, \Phi)}{\partial(x,y)} + Gr^{-2/5}\sigma^{-1}\nabla^2\Phi = 0 \qquad (2)$$

where $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2$. In these equations ψ and Φ are defined such that $(u, v) = (\partial\psi/\partial y, -\partial\psi/\partial x)$, $\Phi = (T-T_\infty)\,Gr^{1/5}/T_r$ and $Gr = g\beta L^3 T_r/\nu^2$ is the Grashof number.

For the present problem, the boundary conditions of equations (1) and (2) are

$$\psi = \frac{\partial\psi}{\partial\theta} = \frac{\partial\Phi}{\partial\theta} = 0 \quad \text{on} \quad \theta = 0, \quad 0 < r < \infty \qquad (3)$$

$$\psi = \frac{\partial\psi}{\partial r} = \Phi = 0 \quad \text{on} \quad \theta = \frac{\pi}{2}, \quad 0 < r < \infty \qquad (4)$$

$$\psi = 0(r), \quad \Phi \to 0 \quad \text{as} \quad r \to \infty, \quad 0 < \theta < \pi/2 \qquad (5)$$

along with the global heat flux condition (see Afzal [5])

$$Gr^{1/5}\left[\sigma\int_0^\infty \frac{\partial\psi}{\partial y}\Phi\,\mathrm{d}y - Gr^{-2/5}\int_0^\infty \frac{\partial\Phi}{\partial x}\,\mathrm{d}y\right] = Q \qquad (6)$$

where $Q = q_s(\mu c_p T_r)^{-1}$. In the following, assuming the Grashof number to be large ($Gr \gg 1$), we shall obtain solutions for the velocity and temperature fields by applying the method of matched asymptotic expansions (see Van Dyke [14]). The method of solution involves dividing the flow field into three district regions: two are the inner regions close to

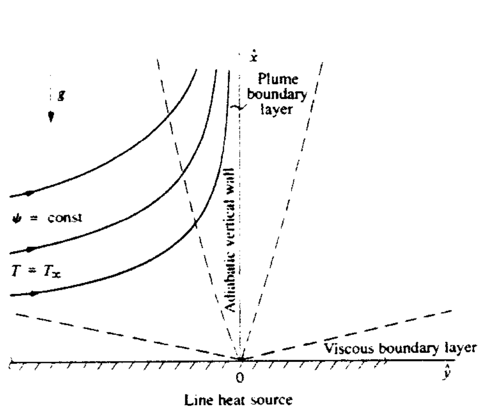

FIG. 1. Physical model and coordinate system.

the walls and the other is the outer region far from the walls. Separate, locally valid expansions of the stream and temperature functions are developed for these three regions.

In the outer region the solution is given by

$$\psi = Gr^{-1/5}[\tilde{\psi}_2(x,y)+Gr^{-1/10}\tilde{\psi}_3(x,y)+\text{h.o.t.}] \quad (7)$$

and

$$\Phi \text{ is exponentially small.} \quad (8)$$

Substituting (7) and (8) into (1) and (2) gives that the first two outer approximations $\tilde{\psi}_2$ and $\tilde{\psi}_3$ are described by the Laplace equation

$$\nabla^2\tilde{\psi}_n = 0, \quad n = 2,3 \quad (9)$$

with the boundary conditions $\tilde{\psi}_n(x,0)$ and $\tilde{\psi}_n(0,y)$ match with the inner expansions at the edges of the boundary layers, and the infinity condition

$$\nabla\tilde{\psi}_n \to 0 \quad \text{as} \quad r\to\infty, \quad 0<\theta<\pi/2. \quad (10)$$

In the inner regions, on the other hand, the solution separates into two distinct forms. In the plume region there is a structure of the solution similar to that discussed in refs. [8, 12], namely

$$\psi = Gr^{-1/5}[\psi_1(x,Y)+Gr^{-1/5}\psi_2(x,Y)+Gr^{-3/10}\psi_3(x,Y)+\text{h.o.t.}] \quad (11)$$

$$\Phi = \Phi_1(x,Y)+Gr^{-1/5}\Phi_2(x,Y)+Gr^{-3/10}\Phi_3(x,Y)+\text{h.o.t.} \quad (12)$$

in which $x(>0)$ and the inner variable $Y(=Gr^{1/5}y)$ are fixed as $Gr\to\infty$. It should be noted here that the third terms in the above expansions represent the effect of the horizontal wall on the development of the plume. Then the inner solution associated with the horizontal viscous boundary layer can be expressed as in ref. [12], i.e.

$$\psi = Gr^{-1/5}[Gr^{-1/10}\bar{\psi}_1(X,y)+Gr^{-1/5}\bar{\psi}_2(X,y)+\text{h.o.t.}] \quad (13)$$

in which $y(>0)$ and the inner variable $X(=Gr^{1/10}x)$ are fixed as $Gr\to\infty$. Perturbation equations are then obtained by collecting like powers of Gr. Employing the usual asymptotic matching technique, boundary conditions are determined for each level of expansion in the inner and outer regions. Thus, after matching it is found that the terms of inner expansions (11)–(13) must satisfy

$$f_1'''+\tfrac{3}{5}f_1f_1''-\tfrac{1}{5}f_1'^2+h_1 = 0$$
$$\sigma^{-1}h_1''+\tfrac{3}{5}(f_1h_1)' = 0$$
$$f_1(0) = f_1'(0) = f_1'(\infty) = h_1'(0) = h_1(\infty) = 0$$
$$\sigma\int_0^\infty f_1'h_1\,\mathrm{d}\eta = Q; \quad (14)$$

$$\bar{f}_1'''-\tfrac{3}{10}\bar{f}_1\bar{f}_1''+\tfrac{2}{5}(\bar{\alpha}_1^2-\bar{f}_1'^2) = 0$$
$$\bar{f}_1(0) = \bar{f}_1'(0) = 0$$
$$\bar{f}_1'(\infty) = \tfrac{3}{5}f_1(\infty)/\sin(3\pi/10) \equiv \bar{\alpha}_1; \quad (15)$$

$$f_2'''+\tfrac{3}{5}f_1f_2''+\tfrac{1}{5}f_1'f_2'+h_2 = 0$$
$$\sigma^{-1}h_2''+\tfrac{3}{5}(f_1h_2'+2f_1'h_2+h_1f_2') = 0$$
$$f_2(0) = f_2'(0) = h_2'(0) = h_2(\infty) = 0$$
$$f_2'(\infty) = -\tfrac{3}{5}f_1(\infty)\cot(3\pi/10) \equiv \alpha_2$$
$$\int_0^\infty (f_1'h_2+f_2'h_1)\,\mathrm{d}\eta = 0; \quad (16)$$

$$\bar{f}_2'''-\tfrac{3}{10}\bar{f}_1\bar{f}_2''+\tfrac{11}{10}(\bar{\alpha}_1\bar{\alpha}_2-\bar{f}_1'\bar{f}_2') = 0$$
$$\bar{f}_2(0) = \bar{f}_2'(0) = 0$$
$$\bar{f}_2'(\infty) = -\tfrac{3}{10}\bar{A}_1\cot(3\pi/20) \equiv \bar{\alpha}_2; \quad (17)$$

$$f_3'''+\tfrac{3}{5}f_1f_3''+\tfrac{1}{2}f_1'f_3'-\tfrac{3}{10}f_1''f_3+h_3 = 0$$
$$\sigma^{-1}h_3''+\tfrac{3}{5}f_1h_3'+\tfrac{3}{2}f_1'h_3+\tfrac{3}{5}h_1f_3'-\tfrac{3}{10}h_1'f_3 = 0$$
$$f_3(0) = f_3'(0) = h_3'(0) = h_3(\infty) = 0$$
$$f_3'(\infty) = \tfrac{3}{10}\bar{A}_1/\sin(3\pi/20) \equiv \alpha_3$$
$$\int_0^\infty (f_1'h_3+f_3'h_1)\,\mathrm{d}\eta = 0; \quad (18)$$

where primes denote differentiation with respect to either η or $\bar{\eta}$. Inspection of equations (14)–(18) indicates that the decay of the velocity field is exponential in the plume layer whereas it is algebraic in the horizontal layer, i.e. the functions $\bar{f}_1$, f_2, $\bar{f}_2$ and f_3 behave as

$$\bar{f}_1(\bar{\eta}) \sim \bar{\alpha}_1\bar{\eta}+\bar{A}_1+O(\bar{\eta}^{-5/3})$$
$$f_2(\eta) \sim \alpha\eta+A_2+O(\exp(-\gamma\eta))$$
$$\bar{f}_2(\bar{\eta}) \sim \bar{\alpha}_2\bar{\eta}+\bar{A}_2+O(\bar{\eta}^{-8/3})$$
$$f_3(\eta) \sim \alpha_3\eta+A_3+O(\exp(-\gamma\eta)) \quad (19)$$

when η and $\bar{\eta}$ tend to infinity. Here $\gamma = 3/5f_1(\infty)$ and the similarity variables η and $\bar{\eta}$ are defined by

$$\eta = Y/x^{2/5} \quad \text{and} \quad \bar{\eta} = X/y^{7/10}. \quad (20)$$

From equation (9) and the matching considerations, the first two terms of the outer expansion (7) must satisfy

$$\nabla^2\tilde{\psi}_2 = 0$$
$$\tilde{\psi}_2(x,0) = f_1(\infty)x^{3/5}, \quad \tilde{\psi}_2(0,y) = 0; \quad (21)$$
$$\nabla^2\tilde{\psi}_3 = 0$$
$$\tilde{\psi}_3(x,0) = 0, \quad \tilde{\psi}_3(0,y) = \bar{A}_1y^{3/10} \quad (22)$$

along with the conditions at infinity (10). Solving equations (21) and (22), we obtain

$$\tilde{\psi}_2 = -f_1(\infty)r^{3/5}\frac{\sin[\frac{3}{5}(\theta-\pi/2)]}{\sin(3\pi/10)}$$
$$\tilde{\psi}_3 = \bar{A}_1r^{3/10}\frac{\sin(3\theta/10)}{\sin(3\pi/20)}. \quad (23)$$

However, the inner expansions (11)–(13) are not unique. To each of them may be added any one of an infinite set of eigensolutions which have the form

$$\psi_k = C_k Gr^{-(1+\lambda_k)/5}x^{3(1-\lambda_k)/5}F_k(\eta)$$
$$\Phi_k = C_k Gr^{-\lambda_k/5}x^{-3(1+\lambda_k)/5}H_k(\eta) \quad (24)$$

and

$$\bar{\psi}_k = \bar{C}_k Gr^{-(3+\bar{\lambda}_k)/10}y^{(3-\bar{\lambda}_k)/10}\bar{F}_k(\bar{\eta}). \quad (25)$$

Here λ_k and $\bar{\lambda}_k$ are the eigenvalues associated with the inner boundary layers while C_k and $\bar{C}_k$ are multiplicative constants, being indeterminate in general. The differential equations for the functions F_k and H_k are

$$F_k'''+\tfrac{3}{5}f_1F_k''-\tfrac{1}{5}(2-3\lambda_k)f_1'F_k'+\tfrac{3}{5}(1-\lambda_k)f_1''F_k+H_k = 0 \quad (26)$$

$$\sigma^{-1}H_k''+\tfrac{3}{5}f_1H_k'+\tfrac{3}{5}(1+\lambda_k)f_1'H_k+\tfrac{3}{5}(1-\lambda_k)h_1'F_k+\tfrac{3}{5}h_1F_k' = 0 \quad (27)$$

with the boundary conditions

$$F_k(0) = F_k'(0) = F_k'(\infty) = H_k'(0) = H_k(\infty) = 0. \quad (28)$$

Numerical integration of equations (26) and (27) subject to the boundary conditions (28) gives the first value of λ_k as $\lambda_1 = 5/3$ for all values of σ. This eigenvalue introduces a term in the inner expansions (11) and (12) which in order of magnitude lies between the third and fourth terms of each series. The next eigenvalue is dependent on the value of σ and is 3.231 for $\sigma = 6.7$. Thus, the assumed form of the solution in expansions (11) and (12) is appropriate to $O(Gr^{-1/2})$ and $O(Gr^{-3/10})$, respectively.

The equation satisfied by $\bar{F}_k$ is

$$\bar{F}_k''' - \tfrac{1}{10}\bar{f}_1\bar{F}_k'' - \tfrac{1}{10}(8+\bar{\lambda}_k)\bar{f}_1'\bar{F}_k' - \tfrac{1}{10}(3-\bar{\lambda}_k)\bar{f}_1''\bar{F}_k = 0 \qquad (29)$$

which has to be solved subject to the boundary conditions

$$\bar{F}_k(0) = \bar{F}_k'(0) = \bar{F}_k'(\infty) = 0. \qquad (30)$$

A numerical inspection of these equations shows that they do not possess a solution for any real $\bar{\lambda}_k > 0$ and therefore the expansion (13) is correct to $O(Gr^{-2/5})$.

The boundary layer characteristics such as skin friction coefficients

$$C_f^{\mathrm{I}} = 2Gr^{-2/5}x^{-2/5}(\partial u/\partial y)_{y=0}, \quad C_f^{\mathrm{II}} = Gr^{-2/5}(\partial v/\partial x)_{x=0}$$

and the adiabatic wall temperature $T_a = (T_w - T_\infty)/T_r$, can now be expressed in terms of the similarity variables as

$$C_f^{\mathrm{I}}/Gr_x^{-1/5} = 2f_1''(0) + 2f_2''(0)\,Gr_x^{-1/5} + 2f_3''(0)\,Gr_x^{-3/10} + \text{h.o.t.} \qquad (31)$$

$$C_f^{\mathrm{II}}/Gr_y^{-1/2} = \bar{f}_1''(0) + \bar{f}_2''(0)\,Gr_y^{-3/10} + \text{h.o.t.} \qquad (32)$$

$$T_a/Gr_x^{-1/5} = h_1(0) + h_2(0)\,Gr_x^{-1/5} + h_3(0)\,Gr_x^{-3/10} + \text{h.o.t.} \qquad (33)$$

where the local Grashof numbers Gr_x and Gr_y are defined as $Gr_x = g\beta T_r\hat{x}^3/\nu^2$ and $Gr_y = g\beta T_r\hat{y}^3/\nu^2$, respectively.

3. RESULTS AND DISCUSSION

The numerical results of the first- and second-order perturbation functions (f_1, h_1) and (f_2, h_2) have been obtained in Afzal [5] for the Prandtl numbers $\sigma = 0.72$ (air) and 6.7 (water). Here we have determined the numerical values of the perturbation functions ($\bar{f}_1, \bar{f}_2$ and f_3, h_3) associated with the contribution of the inner stream functions and temperature due to the presence of the horizontal wall for the same values of σ, using the Runge–Kutta–Merson method and Newton iteration. In doing so, we have recomputed Afzal's solution for (f_1, h_1) and (f_2, h_2). Graphs of these functions are displayed in Figs. 2 and 3 for $\sigma = 0.72$ only. It should, however, be noted that Azfal's results for the functions (f_2, h_2) are inconsistent with the very carefully performed numerical results found in this paper and this discrepancy may be seen by comparing our results, from Figs. 2 and 3, with those from Figs. 5 and 6 in Afzal [5]. Further, a detailed investigation of Afzal's results show that they are themselves inconsistent.

The numerical results for skin friction coefficients and

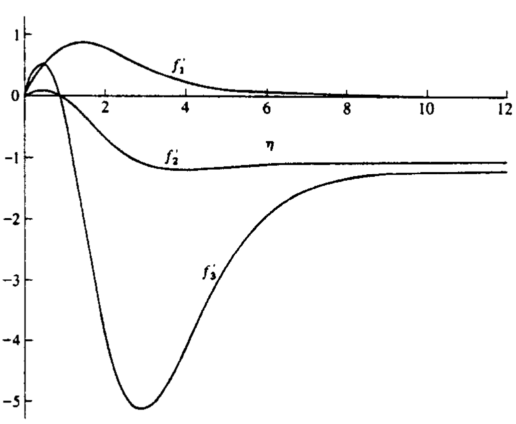

FIG. 2. The velocity functions associated with the plume boundary layer for $\sigma = 0.72$.

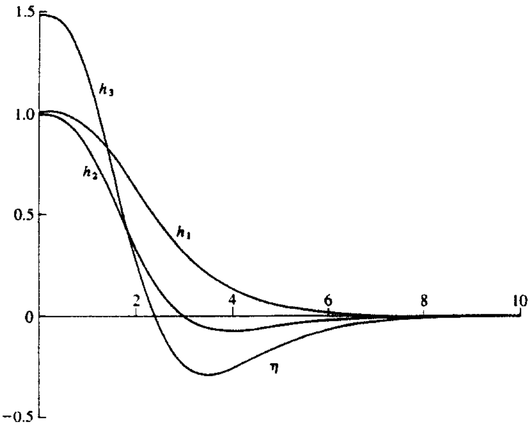

FIG. 3. The temperature functions for $\sigma = 0.72$.

adiabatic wall temperature given by equations (31)–(33) are

$$C_f^{\mathrm{I}}/Gr_x^{-1/5} = \begin{cases} 2.6201 + 0.8761Gr_x^{-1/5} \\ \quad + 5.1859Gr_x^{-3/10} + \text{h.o.t.} & \sigma = 0.72 \\ 1.8596 + 0.1245Gr_x^{-1/5} \\ \quad + 2.3932Gr_x^{-3/10} + \text{h.o.t.} & \sigma = 6.7 \end{cases} \qquad (34)$$

$$C_f^{\mathrm{II}}/Gr_y^{-1/2} = \begin{cases} 1.7479 + 1.9813Gr_y^{-1/10} + \text{h.o.t.} & \sigma = 0.72 \\ 0.5918 + 0.9794Gr_y^{-1/10} + \text{h.o.t.} & \sigma = 6.7 \end{cases} \qquad (35)$$

$$T_a/Gr_x^{-1/5} = \begin{cases} 1 + 0.9875Gr_x^{-1/5} + 5.9166Gr_x^{-3/10} \\ \quad + \text{h.o.t.} & \sigma = 0.72 \\ 1 + 0.3895Gr_x^{-1/5} + 4.3291Gr_x^{-3/10} \\ \quad + \text{h.o.t.} & \sigma = 6.7. \end{cases} \qquad (36)$$

Also, the equivalent line source temperature $T_r = q_s(\mu C_p Q)^{-1}$ is

$$Q = \sigma\int_0^\infty f_1' h_1\,\mathrm{d}\eta = \begin{cases} 1.0915 \text{ for } \sigma = 0.72 \\ 2.6446 \text{ for } \sigma = 6.7. \end{cases} \qquad (37)$$

Again, by inspection of the second-order terms in Afzal's equations (84) and (85), we see that they are not in agreement with those given by equations (34) and (36) in the present paper. We believe our numerical results presented by equations (34)–(36) to be accurate to the number of significant figures shown. Then, we notice from these equations that the skin friction coefficients and the adiabatic wall temperature on the axis of the plume are underpredicted by the first-order boundary layer solution. Further, the third-order correction terms add to the second-order terms in the underprediction of the values. For values of the Grashof number less than $O(10^5)$ it is observed that the errors in using the first-order theory are incurred in excess of the order of 10%. On the horizontal wall the second-order correction reinforces the first-order correction term to further increase the magnitude of the skin friction. This increase in the skin friction coefficients and the adiabatic wall temperature implies a decrease of the thickness of the boundary layers for moderately large values of the Grashof number.

In the expressions (34) and (36) it is observed that the effects of the horizontal wall leads to a correction of $O(Gr^{-3/10})$. The first eigensolution which modifies these results is of $O(Gr^{-1/3})$ and therefore expressions (34)–(36)

are correct to the number of terms quoted. In fact, the eigensolutions form the next correction to these expressions. In the work of Afzal [5] he ignores the effects of the boundary layer which is formed on the horizontal wall and hence his solution technique is only correct up to, and including, his second term. In turn, this leads to smaller corrections to the skin friction and adiabatic temperature on the axis of the plume. In the present study, we have shown that the presence of the horizontal wall substantially changes the third-order boundary layer correction terms. We shall then have an adequate description of the plume flow characteristics.

The fluid flow pattern outside the inner boundary layers is shown in Fig. 4 for $\sigma = 0.72$ with $Gr = 10^{10}$, and similar results have been obtained for other values of the Grashof and Prandtl numbers. All these results show, as expected, that for a given value of the Grashof number the smaller the Prandtl number the more intense is the induced velocity. We also conclude from these figures that at a small distance from the horizontal wall the effect of the viscous layer is to make the streamlines enter the convective boundary layer such that they are convex upwards, whereas in the absence of the horizontal viscous layer the streamlines enter convex downwards. It transpires that, rather than becoming less important, viscosity gets more and more dominant as we move to the outer edges of the inner boundary layers and the outer flow is rotational. It is thus quite conceivable that the complete Navier–Stokes equation has to be solved in the outer region. This is also suggested by the algebraic behaviour of $(\bar{f}_1, \bar{f}_2)$ given by equation (19) when $\bar{\eta}$ is large. This matter has been also discussed by Schneider [15], who pointed out that jets and plumes induce a rotational outer flow. Consequently, we question Afzal's observation that for a plane plume with horizontal bounding surfaces, the outer layer is essentially inviscid.

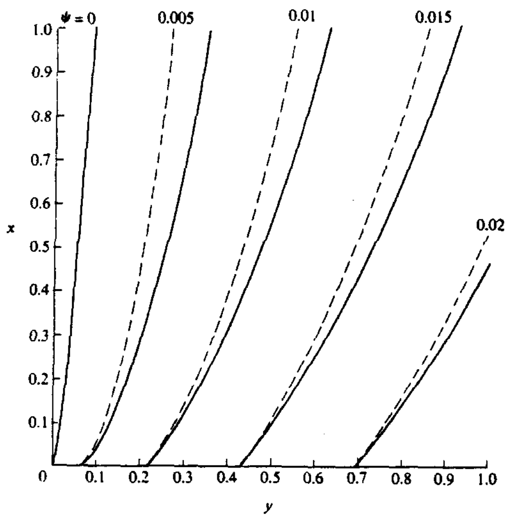

FIG. 4. The streamlines associated with the outer flow at $Gr = 10^{10}$ for $\sigma = 0.72$: (------) one term; (———) two terms.

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